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Approximate solutions of functional equations with small delay using the WKB method

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Abstract. A class of functional equations containing delay terms in which the delay is small compared with an overall scale, or difference equations with slowly varying coefficients, is analysed using the WKB method. A particular example in which a periodic solution is required is analysed in detail. Results analogous to quantization conditions are discussed as are turning points. The approximate solutions obtained by the WKB method are compared with exact solutions where available in simple cases and with numerical integration and excellent agreement is obtained.

1. Introduction

The WKB method [1–3] enables approximate solutions of Schrödinger's equation to be found over a range of values of the independent variable with good accuracy away from turning points. In bound-state eigenvalue problems, connection formulae together with the requirement of bounded solutions at infinity generate the Bohr–Sommerfeld quantization condition for the eigenvalues [4]. Alternative formulations of the WKB method exist [5] as well as generalizations [6] and Liouville–Green approximations provide a more general approach to the approximate solution of equations of the Schrödinger type [7–11].

For delay equations with non-constant coefficients, a variety of approximation methods exist [12], including infinite-series solutions [13, 14] and Fourier series methods for equations with periodic coefficients [15]. The behaviour of a class of delay equations containing a small delay has been studied [16] as well as the existence of periodic solutions of difference equations with periodic coefficients [17].

In this paper we examine a particular class of linear delay equation equivalent to a two- or three-term difference equation. In the two-term case, a formal analytic solution is possible; this case is useful for comparison purposes. In the three-term case, no such analytic solution is possible. Nevertheless, a numerical solution can be generated and compared with the approximate solution derived using the WKB method. Although we study a specific problem in detail, the methods are more generally applicable. The specific problem is useful in that it contains many features of the WKB method of solution, including periodic solutions, eigenvalues and the possibility of turning points. In the next section we begin with some general considerations. Following this we apply the method to the specific problem before widening the discussion and drawing parallels with the solution of differential equations of the Schrödinger type. As is the case with the WKB method when applied to such differential equations, the functional or difference equations studied here are linear and therefore may have the general form $\sum_{n=-N}^{M} f_n(x)\, y(x+n\varepsilon) = 0$. The principal benefit of the method when

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applied to two- or three-term equations (for example, when $N = 0$ and $M = 1$ or 2) is that a substantial part of the analysis can be carried out analytically.

2. Basic equations

In the WKB method for obtaining an approximate solution to the differential equation

$$\varepsilon^2 \frac{d^2 y}{dx^2} = F(x)\, y \tag{1}$$

the form of the solution is taken as

$$y(x) = y(0) \exp\left\{\frac{1}{\varepsilon}\int_0^x [S_0(t) + \varepsilon S_1(t)]\, dt\right\} \tag{2}$$

where the terms in the integrand of (2) are the first two terms in an infinite series truncated to order ε [4]. On substituting (2) into (1) and equating like powers of ε at the zeroth and the first order, we obtain the two solutions

$$y(x) \propto [F(x)]^{-1/4} \exp\left\{\pm\frac{1}{\varepsilon}\int_0^x [F(t)]^{1/2}\, dt\right\}. \tag{3}$$

An alternative approach [18] based on the assumption that $F(x)$ is slowly varying again results in the solutions given in (3). We note here that equation (1) and the form (2) are both singular as $\varepsilon \to 0$ and that, after substituting and cancelling $y(x)$ on both sides of (1), a zeroth-order term results on the left-hand side because the coefficient ε^2 cancels the factor ε^{-2} arising from differentiating (2) twice. Clearly, therefore, if there were an additional term of the form $g(x)\varepsilon\, dy/dx$ in the differential equation, a zeroth-order term would result from this and a more complicated, although still quadratic, equation for S_0 would be found.

Given the above discussion, we can now consider delay terms of the form $y(x + \varepsilon)$, and note that a Taylor expansion in powers of ε results in the infinite series

$$y(x + \varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n y^{(n)}(x)}{n!} \tag{4}$$

where each term has a power of ε which is the same as the order of the derivative. Hence inserting (2) into (4) will give an infinite series in S_0 to order zero. It is not difficult to see that the sum of this series is $\exp S_0$. If the appropriate series for S_1 can also be summed, we can write down corresponding terms up to order ε for $y(x + \varepsilon)$. This is a possible approach, but a simpler alternative is to expand (2) for the corresponding term with a delayed or shifted argument, as follows.

For greater generality, consider $y(x + n\varepsilon)$, where for practical applications $-2 \leqslant n \leqslant 2$. Then from (2)

$$y(x + n\varepsilon) = y(0) \exp\left\{\int_0^{x+n\varepsilon} [S_0(t) + \varepsilon S_1(t)]\, dt\right\}. \tag{5}$$

Hence

$$\frac{y(x + n\varepsilon)}{y(x)} = \exp\left\{\frac{1}{\varepsilon}\int_x^{x+n\varepsilon} [S_0(t) + \varepsilon S_1(t)]\, dt\right\}. \tag{6}$$

Considering the integral in (6) and expanding both $S_0(t)$ and $S_1(t)$ about the point $t = x$, retaining only those terms which will give a result of order ε or ε^2 on integration, we obtain

$$\begin{aligned}\int_x^{x+n\varepsilon} [S_0(t) + \varepsilon S_1(t)]\, dt &= \int_x^{x+n\varepsilon} [S_0(x) + (t - x)S_0'(x) + \varepsilon S_1(x)]\, dt \\ &= n\varepsilon S_0(x) + \varepsilon^2\left\{\tfrac{1}{2}n^2 S_0'(x) + n S_1(x)\right\}.\end{aligned} \tag{7}$$

Dividing (7) by ε and taking the exponential, we obtain the product of two exponentials the second of which has an argument proportional to ε. This too may be expanded up to first order in ε. Hence, the zeroth- and first-order terms of (6) are

$$\frac{y(x+n\varepsilon)}{y(x)} = \left[1+\varepsilon\left\{\tfrac{1}{2}n^2 S_0'(x) + nS_1(x)\right\}\right]\exp\{nS_0(x)\}. \tag{8}$$

We notice now that equations containing only terms proportional to $y(x)$, $y(x+\varepsilon)$ and $y(x+2\varepsilon)$ or $y(x)$, $y(x-\varepsilon)$ and $y(x-2\varepsilon)$ will be quadratic in the quantity $\exp S_0$ and allow complete approximate solution by the WKB method. For example, if

$$y(x) + p(x)y(x-\varepsilon) + q(x)y(x-2\varepsilon) = 0 \tag{9}$$

then

$$\exp 2S_0 + p(x)\exp S_0 + q(x) = 0. \tag{10}$$

We note that if there are turning points of (9), where $p^2 = 4q$, these manifest themselves clearly in the solution for S_0, which contains $(p^2-4q)^{1/2}$, in the same way that those of the differential equation (1) do in the solution (3). In the next section, we study a specific example in detail.

3. Detailed example

The equation to which we shall apply the method outlined above is a model of active mode-locking [19] in which the pulse profile V satisfies the difference equation

$$V_{i+n} = (1-R)g_i V_i + RV_{i+n-1} \tag{11}$$

or equivalently the functional equation

$$V(x+n\varepsilon) = (1-R)g(x)V(x) + RV(x+(n-1)\varepsilon). \tag{12}$$

Here R is a constant mirror reflectivity close to but less than unity, $g(x)$ is the gain factor and $V(x)$ and $g(x)$ are periodic with period 2π. The variable x, $0 \leqslant x \leqslant 2\pi$, measures the normalized position in the cavity and ε is the ratio of the mismatch time (the cavity round-trip time minus the time between pumping pulses) and the round-trip time. The value of n is either zero or ± 1. When $n = 0$ or 1, equation (12) gives a simple relationship between adjacent pulse values and a formal solution containing a product of values of the function g. The function g in this model must be specified but, for all choices, g contains a constant multiplicative factor f, related to the average inversion in the laser medium, which is unknown initially and must be determined from the requirement of a self-reproducing pulse. Hence we may consider f to be an eigenvalue determined by the requirement that $V(2\pi) = V(0)$, making the numerical solution a non-trivial one. For $n = -1$, no straightforward formal solution exists and the eigenvalue problem remains. To indicate the presence of this eigenvalue, we shall write $g_f(x)$. We will first study the simple case $n = 1$ in order to compare the exact and approximate solutions.

When $n = 1$, we have

$$V(x+\varepsilon) = [(1-R)g_f(x) + R]V(x) \tag{13}$$

the solution of which can be written in the form

$$V(n\varepsilon) = V(0)\prod_{k=0}^{n-1}[(1-R)g_f(k\varepsilon) + R] \tag{14}$$

where $n = 1, 2, \ldots, N = 2\pi/\varepsilon$. The condition that $V(2\pi) = V(0)$ then gives the equation

$$\prod_{k=0}^{N-1}[(1-R)g_f(k\varepsilon) + R] = 1 \tag{15}$$

or, equivalently,

$$\sum_{k=0}^{N-1}\ln[(1-R)g_f(k\varepsilon) + R] = 0 \tag{16}$$

for the eigenvalue f. We now compare this with a WKB solution to (13). Dividing (13) by $V(x)$ and using (8) with $n = 1$ we obtain

$$\left[1 + \varepsilon\left\{\tfrac{1}{2}S_0' + S_1\right\}\right]\exp S_0 = (1-R)g_f(x) + R. \tag{17}$$

Hence

$$S_0(x) = \ln[(1-R)g_f(x) + R] \tag{18}$$

and

$$\tfrac{1}{2}S_0' + S_1 = 0. \tag{19}$$

In the WKB solution, we require the term

$$\exp\left\{\int S_1(x)\,\mathrm{d}x\right\} = \exp\{-S_0/2\} = [(1-R)g_f(x) + R]^{-1/2} \tag{20}$$

using (18) and (19). The WKB solution is therefore

$$V(x) = V(0)\left[\frac{(1-R)g_f(0) + R}{(1-R)g_f(x) + R}\right]^{1/2}\exp\left\{\frac{1}{\varepsilon}\int_0^x \ln[(1-R)g_f(t) + R]\,\mathrm{d}t\right\}. \tag{21}$$

The relationship between this approximate solution and (14) can easily be found, as follows. Evaluating the integral in (21) with $x = n\varepsilon$ using the trapezium rule, we find

$$\int_0^{n\varepsilon}\ln[(1-R)g_f(t) + R]\,\mathrm{d}t \approx \tfrac{1}{2}\varepsilon\Bigg\{\ln[(1-R)g_f(0) + R] + \ln[(1-R)g_f(n\varepsilon) + R] + 2\sum_{k=1}^{n-1}\ln[(1-R)g_f(k\varepsilon) + R]\Bigg\} \tag{22}$$

so that from (21)

$$V(n\varepsilon) = V(0)[(1-R)g_f(0) + R]\prod_{k=1}^{n-1}[(1-R)g_f(k\varepsilon) + R] \tag{23}$$

which is the same as (14). On setting $V(2\pi) = V(0)$, the eigenvalue equation for f becomes, within the WKB approximation,

$$\int_0^{2\pi}\ln[(1-R)g_f(t) + R]\,\mathrm{d}t = 0. \tag{24}$$

The required value of f must therefore be such that the integrand changes sign. Since $g_f(t)$ is periodic with period 2π and resembles a simple sinusoid, this integrand usually has two sign changes in the range of integration. Similar results to those above are obtained for the case $n = 0$ of (12).

When $n = -1$, equation (12) becomes

$$V(x-\varepsilon) = (1-R)g_f(x)V(x) + RV(x-2\varepsilon). \tag{25}$$

Dividing by $V(x)$ and using (8) with $n = -1$ and -2 gives

$$[1+\varepsilon\{\tfrac{1}{2}S_0' - S_1\}]\exp\{-S_0\} = (1-R)g_f(x) + R[1+\varepsilon\{2S_0' - 2S_1\}]\exp\{-2S_0\}. \tag{26}$$

Equating like orders of ε, we obtain the equations

$$R\exp\{-2S_0\} - \exp\{-S_0\} + (1-R)g_f(x) = 0 \tag{27}$$

and

$$\tfrac{1}{2}S_0' - S_1 = 2R(S_0' - S_1)\exp\{-S_0\}. \tag{28}$$

Hence we obtain from (27) the two solutions

$$S_0(x) = -\ln\left[\frac{1}{2R}\left\{1 \pm \sqrt{1-4R(1-R)g_f(x)}\right\}\right]. \tag{29}$$

Solving (28) for S_1, we again find that the integral of S_1 can be calculated in terms of S_0. Explicitly,

$$\int^x S_1(t)\,\mathrm{d}t = S_0(x) - \tfrac{1}{2}\ln|\exp S_0(x) - 2R| \tag{30}$$

so that

$$\exp\left\{\int^x S_1(t)\,\mathrm{d}t\right\} = \frac{\exp S_0(x)}{|\exp S_0(x) - 2R|^{1/2}}. \tag{31}$$

This term can be expressed in terms of $g_f(x)$ and the resulting solutions of (25) are found from (2) to be

$$V(x) = V(0)\left[\frac{1\pm\sqrt{1-4R(1-R)g_f(0)}}{1\pm\sqrt{1-4R(1-R)g_f(x)}}\right]^{1/2}\left[\frac{1-4R(1-R)g_f(0)}{1-4R(1-R)g_f(x)}\right]^{1/4}$$
$$\times\exp\left\{-\frac{1}{\varepsilon}\int_0^x \ln\left[\frac{1}{2R}\left\{1\pm\sqrt{1-4R(1-R)g_f(t)}\right\}\right]\mathrm{d}t\right\}. \tag{32}$$

Note that the solution diverges at the turning points where $1 - 4R(1-R)g_f(x) = 0$ and that these points may or may not exist depending on the value of f. Assuming there are no turning points, then the requirement that $V(2\pi) = V(0)$ results in two possible equations for f, since

$$\int_0^{2\pi} \ln\left[\frac{1}{2R}\left\{1\pm\sqrt{1-4R(1-R)g_f(t)}\right\}\right]\mathrm{d}t = 0. \tag{33}$$

However, it is straightforward to show that if R is close to unity, the minus sign in (33) yields an integrand that is always negative allowing no solution to (33). Hence the periodic solution is that with the plus sign in (32) and (33) so that f is determined by

$$\int_0^{2\pi} \ln\left[\frac{1}{2R}\left\{1+\sqrt{1-4R(1-R)g_f(t)}\right\}\right]\mathrm{d}t = 0. \tag{34}$$

4. Results

We now show results obtained for $n = \pm 1$ using the formulae (21), (24), (32) and (34). The specific gain function chosen is $g_f(x) = f \exp(m \sin x)$, where f is the eigenvalue and m is the modulation index [19]. We first set R and m and use (24) and (34) to perform a simple iteration to find the corresponding value of f. We then plot the function $V(x)$ against x for each set of n, R, m and f values. Throughout, Simpson's integration rule is used with a step size equal to the value of ε, here 0.01. Figure 1(a) displays $V(x)$ for $n = 1$ and $R = 0.95$ with $m = 0.5$ (full curve), $m = 1$ (broken) and $m = 2$ (dotted). Figure 1(b) shows the function $V(x)$ with the same value of R and values of m for $n = -1$. In figure 2 we show the variation in $V(x)$ with R choosing $m = 0.5$. Figure 2(a) displays $V(x)$ for $n = 1$ with $R = 0.8$ (full), $R = 0.9$ (broken) and $R = 0.95$ (dotted). Figure 2(b) shows $V(x)$ for the same m and R values and $n = -1$.

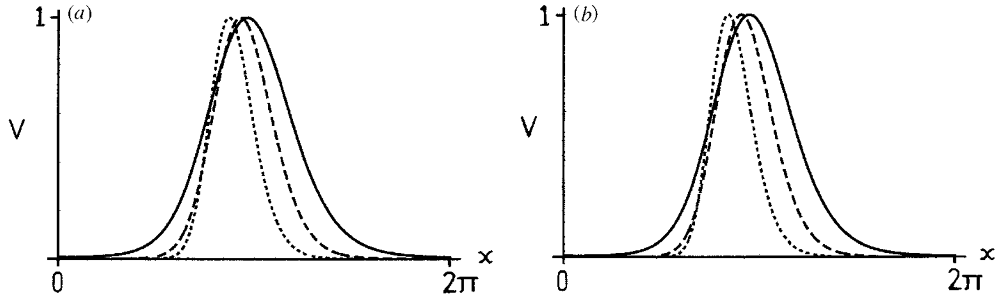

Figure 1. The function $V(x)$ normalized to unit height with $R = 0.95$, $\varepsilon = 0.01$, $m = 0.5$ (full curve), $m = 1$ (broken curve) and $m = 2$ (dotted curve) for (a) $n = 1$ and (b) $n = -1$.

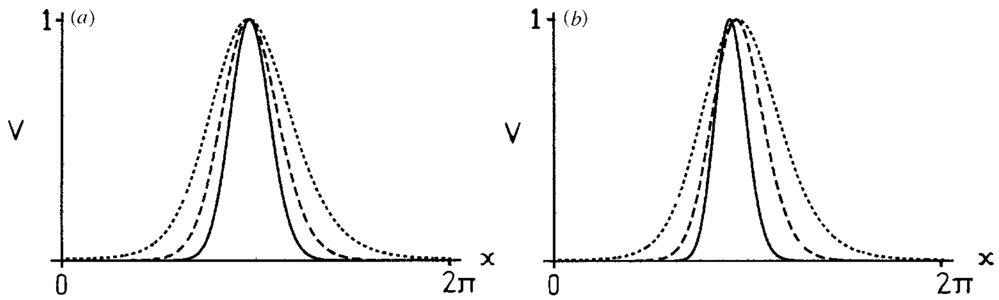

Figure 2. The function $V(x)$ normalized to unit height with $m = 0.5$, $\varepsilon = 0.01$, $R = 0.8$ (full curve), $R = 0.9$ (broken curve) and $R = 0.95$ (dotted curve) for (a) $n = 1$ and (b) $n = -1$.

A comparison was made between these results and a numerical solution of (13) and (25). Both are 'shooting problems' for which the value of f is unknown at the start but must be found so that $V(N\varepsilon) = V(0)$. The case of $n = 1$ is more straightforward in that a value of f can be chosen, $V(N\varepsilon)$ found by iteration given a value $V(0)$, and then f adjusted until $V(N\varepsilon) = V(0)$. The case of $n = -1$ is more difficult since to begin the iteration both $V(0)$ and $V(\varepsilon)$ must be specified together with the value of f, and hence there are two parameters which must be adjusted until $V(N\varepsilon) = V(0)$. In fact, the values of f calculated using the WKB approximation were used as the basis for the numerical work to reduce the computation time considerably. When the function $V(x)$ is plotted in each case, no visible difference between the numerical and WKB solutions were found. We further studied the comparison between numerical and WKB solutions for the same m and R values, while changing the value of ε from 0.01 to 0.1. Small differences of the order of 1% between solutions could be detected for $\varepsilon = 0.1$ consistent with the terms of order ε^2 being neglected in the WKB solutions. We note that the Simpson's rule integrations employed in evaluating the WKB solutions have an accuracy of order ε^4.

5. Discussion

A further connection may be made between the three-term difference equation (11) with $n = -1$

$$V_{i-1} = (1 - R)g_i V_i + R V_{i-2} \tag{35}$$

and a differential equation of the type (1). Changing V_i in (35) to

$$V_i = y_i \left(\frac{1}{2(1 - R)} \right)^{i-1} \prod_{j=0}^{i-2} \frac{1}{g_{j+2}} \tag{36}$$

we obtain [20] the difference equation

$$y_{i+2} - 2y_{i+1} + y_i = (1 - 4R(1 - R)g_{i+1})y_i. \tag{37}$$

Recognizing the left-hand side as a divided difference for the second derivative we have the same form $\varepsilon^2 y'' = F(x)\, y$ as in (1) if the spacing between successive values V_i is ε. An alternative approach would be to use the WKB solution of (1) together with the transformation (36).

Although the problem in section 3 has a finite range, a general three-term difference equation could be solved using the WKB approximation if the coefficients were slowly varying. For example, if g_i contained only $i\varepsilon$, where $\varepsilon \ll 1$, then the new independent variable becomes $t = i\varepsilon$ and V_{i+1} becomes $V(t + \varepsilon)$. It is also straightforward to show that the exact solutions of a three-term difference equation with constant coefficients are retrieved using the WKB method which truncates in this simple case.

A rather more naive approach may be imagined to the solution of particular cases of the equations studied in this paper. Consider, for example, equation (13) and an approximate solution obtained by Taylor expansion of $V(x + \varepsilon) \approx V(x) + \varepsilon V'(x)$. We then obtain

$$\varepsilon V' = (1 - R)(g_f(x) - 1)V \tag{38}$$

which can readily be integrated. However, if V and V' are both of order unity, and for arbitrary R, a consistent solution is obtained only if $g_f(x) - 1$ remains of order ε. For the particular form of $g_f(x)$ chosen above, this restricts the value of m. In fact, the solution of (38) can readily be shown to be the expansion of the WKB solution for small m. The WKB solution does not suffer from this restriction since it includes relevant contributions from all derivatives of V.

For the range of values of parameters considered in this paper, the question of turning points has not arisen. For non-periodic solutions, the existence of turning points would require the construction of connection formulae and substantial further work would be needed to completely solve the problem. The principal benefit of the WKB solutions, a closed-form expression which is straightforward to calculate, may be lost in that case.

In principal, the technique described in this paper could be applied to functional-differential equations with a true derivative term, with the usual requirement of the WKB method that in the equation to be solved an nth-order derivative term must be multiplied by ε^n. However, it is easily seen that, for example, a term $\varepsilon y'(x)$ generates $S_0 + \varepsilon S_1$ from (2), whereas a term $f(x)\, y(x+\varepsilon)$ generates, from (8), $f(x)[1 + \varepsilon\{S_0'/2 + S_1\}] \exp S_0$, resulting in a transcendental equation for S_0. Again, the benefit of a straightforward analytic solution is lost.

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